Skewness in Lévy Markets

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Lecture IV. PASI - Guanajuato - June 2010[1]

[1] Joint work with José Fajardo Barbachan

Outline

Aim of the talk Understand the shape of the implied volatility curve (or smile) in terms of the parameters of a Lévy process.

Lévy market

Has two assets:

- A deterministic savings account:

$$B_t = e^{rt}, \qquad r \geq 0,$$

- A random stock

$$S_t = S_0 e^{X_t}, \qquad S_0 > 0,$$

where $\{X_t\}$ is a Lévy process under a certain probability **P**.
S pays dividends at constant rate $\delta \geq 0$.

Observe: As

$$X_t = \sigma W_t + (r - \sigma^2/2)t$$

is a Lévy process, we generalize Black Scholes model.

Lévy-Khinchine formula

For $v \in \mathbb{R}$,

$$\mathbf{E}\, e^{ivX_t} = \exp\left(t\psi(iv)\right)$$

where

$$\psi(z) = az + \frac{(\sigma z)^2}{2} + \int_{\mathbb{R}} (e^{zy} - 1 - zh(y))\Pi(dy)$$

Here:

- $h(y) = y\mathbf{1}_{\{|y|<1\}}$ is a truncation function
- a is a real constant
- $\sigma \geq 0$ is the variance of the gaussian part
- Π such that $\int (1 \wedge y^2)\Pi(dy) < +\infty$ is the jump measure

Examples:

- Brownian motion:

$$\psi(z) = \frac{(\sigma z)^2}{2}$$

- Poisson Process with jump magnitude c:

$$\psi(z) = \lambda(e^{cz} - 1),$$

- Sum $X_1 + X_2$ of independent LP:

$$\psi_1(z) + \psi_2(z).$$

Option pricing and implied volatility

Let us recall BS Formula:

$$V(S_0, T) = S_0 \Phi(x_+) - e^{-rT} K \Phi(x_-)$$

with

$$x_\pm = \left(\log \frac{S_0 e^{rT}}{K} \pm \frac{1}{2}\sigma^2 T \right) / (\sigma\sqrt{T}).$$

In fact the price is $V(S_0, K, T, r, \sigma)$ depending on five parameters:

- Three quantities writen in the contract:
  - S_0 the spot price of the stock (i.e. today's price)
  - K the strike or excercise price
  - T the expiration, or excercise time.

Market dependent parameters

- the interest rate in the market r considered "observable" and obtained usually from US bonds with the same expiration
- the volatility σ

Implied volatility Empirical practice shows that option prices do not follow BS model. Despite lots of reasons, to still use BS formula given a price QP, based on the fact that V is increasing in σ (i.e. $\partial V/\partial\sigma > 0$) people solve the equation

$$V(\sigma) = QP,$$

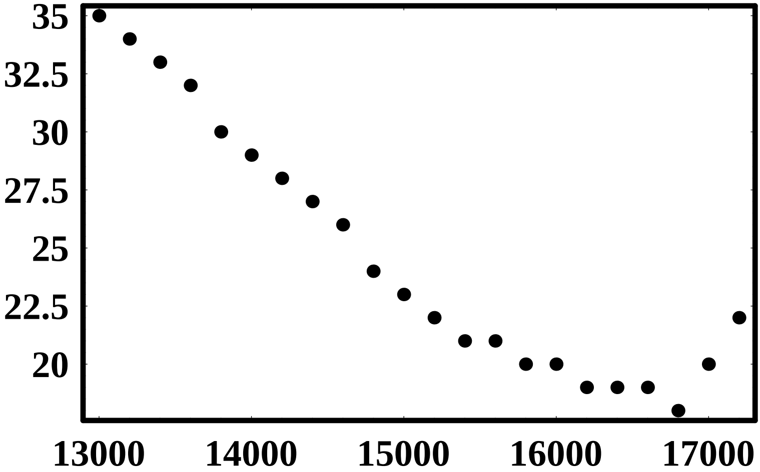
to obtain a value of σ called the volatility *implied* by the option price QP.

An example of implied volatility

In the next slide we have a plot of implied volatility for the Hong Kong stock exchante market. The asset is the *Hang Seng Index* (HSI).

- Date: June 15, 2006.
- Spot price $S_0 = 15248$ (HK dollars)
- Strikes from 13000 to 17200 every 200 HK dollars.
- $r = 0.025$ taken from futures on the HSI.
- $T = 32/247$: 247 trading days, 32 days to expiration.

Volatility Smile for the HSI



Esscher transform

We assume that X is also a Lévy process under the *risk neutral* probability $\mathbf{Q}$. We furthermore assume that

$$d\left(\mathbf{P} \mid \mathcal{F}_t\right) = e^{\theta X_t - \psi(\theta) t} d\left(\mathbf{Q} \mid \mathcal{F}_t\right),$$

i.e. $\mathbf{P}$ and $\mathbf{Q}$ are equivalent measures, and the Radon-Nykodim derivative can be expressed as an exponential of the Lévy process (the so called "log-stock price" process). This change of measure is known as the *Esscher transform*.

- $\mathbf{Q}$ is the probability used to price derivatives.
- $e^{X_t - (r-\delta)t}$ is a $\mathbf{Q}$-martingale.

Duality of eurpean option prices

The price of a call option is

$$\begin{aligned} Call(S_0, K, r, \delta, \psi_{\mathbf{Q}}) &= \mathbf{E}_{\mathbf{Q}}\, e^{-rT}(S_0 e^{X_T} - K)^+ = \\ &= \mathbf{E}_{\mathbf{Q}}(e^{X_T} e^{(\delta - r)T}) e^{-\delta T}(S_0 - K e^{-X_T})^+ \\ &= \mathbf{E}_{\tilde{\mathbf{Q}}}\, e^{-\delta T}(S_0 - K e^{\tilde{X}_T})^+ \\ &= Put(K, S_0, \delta, r, \psi_{\tilde{\mathbf{Q}}}) \end{aligned}$$

the price of a Put option! But we changed

- $S_0 \leftrightarrow K$, $r \leftrightarrow \delta$
- $\mathbf{Q} \to \tilde{\mathbf{Q}}$ the *dual* measure.

Duality of american option prices

Introduce the optimal stopping rules:

$$\tau_c^* = \inf\{t \geq 0 \colon S_t \geq \mathcal{B}_c(t)\} \wedge T\,,$$
$$\tau_p^* = \inf\{t \geq 0 \colon S_t \leq \mathcal{B}_p(t)\} \wedge T\,.$$

where $\mathcal{B}_c$ and $\mathcal{B}_p$ are the *optimal stopping boundaries*
Theorem Assume $\delta > 0$ and $r > 0$. Then

$$C_{am}(S_0, K, r, \delta, T, \psi) = P_{am}(K, S_0, \delta, r, T, \tilde{\psi}),$$

The boundaries satisfy:

$$\mathcal{B}_c(t)\mathcal{B}_p(t) = S_0 K\,.$$

Remark: For Itô processes see Detemple (2001)

Duality for perpetual american options

The optimal stopping rules are

$$\tau_c^* = \inf\{t \geq 0 \colon S_t \geq S_c^*\},$$
$$\tau_p^* = \inf\{t \geq 0 \colon S_t \leq S_p^*\}.$$

Theorem If $\delta > 0$ and $r > 0$, we have

$$C_{perp}(S_0, K, r, \delta, T, \psi) = P_{perp}(K, S_0, \delta, r, T, \tilde{\psi}),$$

And the optial stopping levels satisfy

$$S_c^* S_p^* = S_0 K.$$

Dual market

Is X a Lévy process under $\tilde{\mathbf{Q}}$? We know

$$d\left(\tilde{\mathbf{Q}} \mid \mathcal{F}_t\right) = e^{X_T} e^{(\delta-r)t} d\left(\mathbf{Q} \mid \mathcal{F}_t\right),$$

and we obtain that

$$e^{\psi_{\tilde{\mathbf{Q}}}(z)} = \mathbf{E}_{\tilde{\mathbf{Q}}}\, e^{z\tilde{X}_1} = \mathbf{E}\, e^{X_1} e^{(\delta-r)} e^{-zX_1}$$

$$= \mathbf{E}_{\mathbf{Q}}\, e^{(\delta-r)} e^{(1-z)X_1} = e^{\psi_{\mathbf{Q}}(1-z)-\psi_{\mathbf{Q}}(1)}$$

Concluding[2]

$$\psi_{\tilde{\mathbf{Q}}}(z) = \psi_{\mathbf{Q}}(1-z) - \psi_{\mathbf{Q}}(1).$$

[2] Fajardo-Mordecki (2003, 2006)

The *Dual Market* is a Lévy market with:

- A deterministic savings account:

$$B_t = e^{\delta t}, \qquad \delta \geq 0,$$

- A random stock

$$\tilde{S}_t = K e^{\tilde{X}_t}, \qquad \tilde{S}_0 = K > 0,$$

  where $\{\tilde{X}_t\}$ is a Lévy process under a **risk neutral** probability $\tilde{\mathbf{Q}}$.

- $\tilde{S}$ pays dividends with constant rate $r \geq 0$

Question: Is this dual market *artificial*?

- mm ...
- Detemple (2001) calls it the “auxiliary” market.
- $\tilde{S}$ is the price of KS_0 dollars in stock unities, its a *change of numeraire*, as in Schroder (1999).
- In forex markets it is even more natural: Grabbe (1983) proposes:
  - S is the price of S_0 euros in US dollars
  - $\tilde{S}$ is the price of K US dollars in euros
  - A call to buy S_0 euros at a strike price of K dollars has the same price of a put to buy K dollars with a strike of S_0 euros.

Symmetry

Observe: In Black Scholes model, the law of the discounted and reinvested stock (DRS) under the risk neutral and the dual measure coincide, because $-W$ is a also Wiener process, and the DRS is a martingale.
Question:

- When the effect of the jumps is such that this *symmetry* property hold for Lévy markets?
- Which are the jump distributions that preserve the symmetry?

Merton Model

Consider Merton Jump Diffusion model: i.e

$$\Pi(dy) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)} dy,$$

Bates (1997) finds that this symmetry hold true if

$$2\mu + \delta^2 = 0.$$

But: what happens for Lévy markets?

For Lévy markets, as

$$\psi_{\tilde{\mathbf{Q}}}(z) = \psi_{\mathbf{Q}}(1-z) - \psi_{\mathbf{Q}}(1)$$

we impose

$$\begin{aligned}\psi_{\mathbf{Q}}(z) - (r-\delta) &= \psi_{\tilde{\mathbf{Q}}}(z) - (\delta - r) \\ &= \psi_{\mathbf{Q}}(1-z) - \psi_{\mathbf{Q}}(1) - (\delta - r),\end{aligned}$$

and obtain[3] the *symmetry condition* in Lévy markets is:

$$\Pi_{\mathbf{Q}}(dy) = e^{-y}\Pi_{\mathbf{Q}}(-dy).$$

that generalizes Bates (1997) result for Lévy markets, and answers a question raised by Carr and Chesney (1996)

[3]Details in Fajardo-M. (2003, 2006)

Symmetry condition on the density of the jump measure

Assume that our Lévy measure can has a density:

$$\Pi_{\mathbf{Q}}(dy) = \pi(y)dy.$$

Our symmetry condition then is

$$\pi(y)dy = \Pi_{\mathbf{Q}}(dy) = e^{-y}\Pi_{\mathbf{Q}}(-dy) = e^{-y}\pi(-y)dy,$$

concluding that the symmetry condition is

$$\pi(y) = e^{-y}\pi(-y).$$

Symmetry in Merton model

The condition is

$$\lambda \frac{1}{\delta\sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)} = e^{-y} \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-(-y-\mu)^2/(2\delta^2)}$$

that equating the exponets gives

$$(y - \mu)^2 = 2\delta^2 y + (y + \mu)^2.$$

The quadratic terms cancels, giving

$$-2\mu y = 2\delta^2 y + 2\mu y,$$

that gives Bate's condition:

$$2\mu + \delta^2 = 0, \quad \text{or} \quad \frac{\mu}{\delta^2} = -\frac{1}{2}.$$

Corollary

Under the symmetry condition the volatility smile curve is symmetric in the log-moneyness $m = \log(K/F)$, with

$$F = e^{(r-\delta)T} S,$$

the futures price.

Proof First observe that, due to the put call parity, the implied volatiliy can either be determined equating the call prices or the put prices. Introduce

$$X_t^o = X_t - (r - \delta)t.$$

Condition $S_t / e^{(r-\delta)t}$ is a **Q**-martingale, implies $E_{\mathbf{Q}} e^{X_T^o} = 1$. The duality condition in terms of ψ^o is

$$\psi^o(z) = \tilde{\psi}^o(z).$$

Now we introduce m in our formulas.

$$
\begin{aligned}
Call(S_0, K, r, \delta, \psi_{\mathbf{Q}}) &= e^{-rT}\, \mathbf{E}_{\mathbf{Q}}(S_0 e^{X_T} - K)^+ = \\
&= e^{-rT}\, \mathbf{E}_{\mathbf{Q}}(S_0 e^{X_T^o + (r-\delta)T} - K)^+ \\
&= e^{-rT} F\, \mathbf{E}_{\mathbf{Q}} \left(e^{X_T^o} - K/(S_0 e^{(r-\delta)T}) \right)^+ \\
&= e^{-rT} F\, \mathbf{E}_{\mathbf{Q}}(e^{X_T^o} - e^m)^+ \\
&= e^{-rT} F \times C_1(m, \psi^o).
\end{aligned}
$$

Similar computations show that

$$Put(K, S_0, \delta, r, \psi_{\mathbf{Q}}) = e^{-rT}\, \mathbf{E}_{\mathbf{Q}}(S_0 - Ke^{X_T})^+ =$$

$$\cdots$$

$$= e^{-rT} K\, \mathbf{E}_{\mathbf{Q}}(e^{-m} - e^{-X_T^o})^+$$

$$= e^{-rT} F\; P_1(m, \tilde{\psi}^o)$$

$$= e^{-rT} F\; P_1(m, \psi^o),$$

the last equality by the symmetry condition.

Conclussion

Finally, if $\sigma_i = \sigma_i(m)$ is the implied volatilty for the call option and a log-moneyness m, we have

$$F\ C_1(m, \psi^o) = K\ P_1(-m, \psi^o),$$

$$F\ C_1\left(m, \frac{\sigma_i^2}{2}(z^2 - z)\right) = K\ P_1\left(-m, \frac{\sigma_i^2}{2}(z^2 - z)\right).$$

As the l.h.s terms of this equation coincide, according to the definition of the implied volatility, the r.h.s. coincide too, giving that the implied volatility (determined by the puts) for $-m$ is the same:

$$\sigma_i(m) = \sigma_i(-m)$$

Skewness in Lévy Markets

We propose to consider a Lévy market with jump structure of the form:

$$\Pi(dy) = e^{\beta y} p(y) dy,$$

where $p(y) = p(-y)$. Most models satisfy this assumption. If we apply the symmetry condition in Lévy markets, we came to

$$\beta = -\frac{1}{2}.$$

In this model we can quantify the deviation from symmetry.

We observe:

- In ForeX markets we expect $\beta + 1/2 \sim 0$.
- For stocks and indices we generally have $\beta + 1/2 << 0$.
- There are some exceptions.

Further research

- Robustness of β across different models (CGMY, Meixner, Merton, Hyperbolic, etc.)
- Asymmetry premia: quantify movements of prices according to movements of β.
- Time dependent $\beta(t)$, as longer maturities suggest more symmetric markets.

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